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On the Combination of Independent Two Sample Tests of a General Class

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ERRATA

Title - ON THE COMBINATION OF INDEPENDENT TWO SAMPLE TESTS OF A GENERAL CLASS

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CORRECTIONS:

Page 3: Line 9

From: $y_i n_i$

To: y_{in_i}

Page 4: Line 10

From: $V_i^{(i)}$

To: $V_1^{(i)}$

Page 5: Line 11 (Assumption 2)

From:

$$\int_{I_{N_i}} [J_N(H_{N_i}) - J(H_{N_i})] dS_{m_i}^{(i)}(x) = O_p(1/N_i 1/2)^3; \; i=1,\ldots,k$$

To:

$$\int_{I_{N_i}} \left[J_N(H_{N_i}) - J(H_{N_i})\right] dS_{m_i}^{(i)}(x) = o_p\left(\frac{1}{\sqrt{N_i}}\right) ; \; i=1,\ldots,k$$

Page 5: Line 13 (Assumption 3)

From: $J_{N_i}(1) = O(\sqrt{N_i})$

To: $J_{N_i}(1) = o(\sqrt{N_i})$

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On the Combination of Independent Two Sample Tests of a General Class[1]

M. L. Puri

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1. Summary

Let $X_{i1},\ldots,X_{im_i}$ and $Y_{i1},\ldots,Y_{in_i}$; $i=1,\ldots,k$ be k pairs of samples of mutually independent observations from continuous distribution functions $F_i(x)$ and $G_i(y)$ respectively; $i=1,\ldots,k$. Then for testing the hypothesis $F_i = G_i$; $i=1,\ldots,k$; test statistics of the form (i) $T = \sum_{i=1}^{k} c_i t_i$ and (ii) $Q = \sum_{i=1}^{k} c_i Q_i$ are considered. Here c_i are the weights which may depend upon the sample sizes, t_i student's t statistic for testing the equality of means between two normal populations with the same variance corresponding to the i^{th} pair of samples and Q_i is the Chernoff-Savage Statistic[2] (1958) for the i^{th} pair of samples. Under suitable assumptions, the weights c_i which maximize the local asymptotic powers of the tests (i) and (ii) are obtained. These results are specialized to (a) Pitman's shift alternatives, (b) Lehmann's distribution free alternatives and (c) contaminated alternatives. Finally, the asymptotic efficiences of Q test relative to some of its parametric as well as non-parametric competitors against the above mentioned alternatives are discussed.

2. Introduction

It frequently happens that several independent test statistics are available for testing the same null hypothesis. These may have arisen from several sets of independent samples which cannot be combined perhaps because they are reported by different investigators or because they have not all been gathered under the same conditions. In such situtations, it is often considered reasonable

to combine the various results into a single measure on which an objective judgment of the evidence as a whoce can be based. One measure is advanced by Fisher (1932). He proposed as a test statistic the product of the tail errors of the individual tests. It turns out that -2 times the logarithm of this product has a chi-square distribution with 2k degrees of freedom when the null hypothesis is true, k being the number of tests. For detailed discussion about Fisher's method, the reader is referred to the paper of Wallis (1942). General discussion of combining independent tests can also be found in Birnbaum (1954) and Pearson (1938). Recently, an interesting technique was advanced by Ph. van Elteren (1960). He analyses a class of tests based on linear combinations $\sum_{i=1}^{k} c_i W_i$ of test statistics $W_1, \ldots, W_k$ of k independent two sample Wilcoxon tests. He considers in particular two special linear combinations, when (i) $c_i = c/m_i n_i$ and (ii) $c_i = c/(m_i+n_i+1)$ where c is a positive real number and m_i, n_i are the sample sizes of of the i^{th} set and shows that the test (i) has a region of consistency independent of sample sizes and the test (ii) has asymptotically the maximum power. In this paper, we consider a similar problem in a more general frame work which includes as a special case the problem considered by Ph. van Elteren (1960), mentioned above. Precisely, we consider the following problem.

3. Problem

Let $\underline{X}_i$, $\underline{Y}_i$; i=1,...,k be k pairs of independent stochastic variables about whose cumulative distribution functions, nothing

$$H_i(x) = \lambda_i F_i(x) + (1-\lambda_i)G_i'(x).$$

Let $Z_{N,j}^{(i)} = 1$, if the j^{th} smallest observation in the combined sample of the i^{th} pair comes from $\underline{X_i}$ and otherwise let $Z_{N,j}^{(i)} = 0$. Then the Chernoff-Savage statistic (1958) for the i^{th} pair of samples is

$$(4.1) \qquad Q_i = \frac{1}{m_i} \sum_{j=1}^{N_i} E_{N,j}^{(i)} Z_{N,j}^{(i)}$$

where the $E_{N,j}^{(i)}$ are given numbers. Note that Wilcoxon's statistic for the i^{th} pair of samples is obtained from (4.1) by letting $E_{N,j}^{(i)} = j/N_i$ and the normal score statistic for the corresponding samples by letting $E_{N,j}^{(i)} = E(V_j^{(i)})$ where $V_1^{(i)} < \dots < V_{N_i}^{(i)}$ is an ordered sample of size N_i from a Standard normal distribution.

Following Chernoff-Savage (1958), we shall use the following equivalent form of Q_i:

$$(4.2) \qquad Q_i = \int_{-\infty}^{+\infty} J_N(H_{N_i}(x))dS_{m_i}^{(i)}(x); \; i=1,\dots,k,$$

where $E_{N,j}^{(i)} = J_N(j/N_i)$.

While J_N need be defined only at $1/N_i,\dots,N_i/N_i$ but may have its domain of definition extended to $(0,1]$ by letting J_N be constant on $(j/N_i, (j+1)/N_i]$.

In this paper, we consider the statistics of the form

$$(4.3) \qquad Q = \sum_{i=1}^{k} c_i Q_i$$

where the c's are real positive numbers and may depend upon the sample sizes.

We may test the hypothesis H_o: $F_i(x) = G_i(x)$; $i=1,\ldots,k$ by means of the critical region $Q \geq Q_\alpha$ where Q_α is given by

$$(4.4) \qquad P_{H_o}(Q \geqq Q_\alpha) = \alpha$$

where α is the level of significance. If the distribution of Q under H_o is symmetric with respect to the origin, then the corresponding left-sided test will have a critical region: $Q \leqq -Q_\alpha$ and the two sided test will have a critical region $|Q| \geqq Q_{\alpha/2}$.

5. <u>General Properties of the distribution of Q</u>

In what follows, we make the following assumptions:

(1) $J(H) = \lim_{N\to\infty} J_N(H)$ exists for $0 < H < 1$ and is not constant.

(2) $\int_{I_{N_i}} [J_N(H_{N_i}) - J(H_{N_i})]dS^{(i)}_{m_i}(x) = o_p(1/N_i1/2)^3$; $i=1,\ldots,k$

where $I_{N_i} = \left\{x: 0 < H_{N_i}(x) < 1\right\}$.

(3) $J_{N_i}(1) = O(\sqrt{N_i})$

(4) $|J^{(r)}(H)| = |\frac{d^rJ}{dH^r}| \leqq K\ [H(1-H)]^{-r-(1/2)+\delta}$ for $r = 0,1,2$

and for some $\delta > 0$ and some K.

Then, the application of Chernoff-Savage theorem (1958) yields

$$(5.1) \qquad \lim_{N_i\to\infty} P\left(\frac{Q_i-\mu_i(\theta)}{\sigma_i(\theta)} \leqq x\right) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt,$$

where

$$(5.2) \qquad \mu_i(\theta) = \int_{-\infty}^{+\infty} J[H_i(x)]dF_i(x)$$

and

$$(5.3)\quad N_i\sigma_i^2(\theta) = 2(1-\lambda_i)\Big\{\iint_{-\infty<x<y<\infty} G_i(x)[1-G_i(y)]J'[H_i(x)]J'[H_i(y)]\,dF_i(x)dF_i(y)$$

$$+\frac{(1-\lambda_i)}{\lambda_i}\iint_{-\infty<x<y<\infty} F_i(x)[1-F_i(y)]J'[H_i(x)]J'[H_i(y)]\,dG_i(x)dG_i(y)\Big\}$$

provided $\sigma_i(\theta) \neq 0$.

Thus

$$(5.4)\qquad \mu(0) = E_{H_o}(Q) = \sum_{i=1}^{k} c_i a_i$$

where

$$(5.5)\qquad a_i = \int_{-\infty}^{+\infty} J[F_i(x)]dF_i(x)$$

$$(5.6)\qquad \sigma^2(0) = \mathrm{var}_{H_o}(Q) = \sum_{i=1}^{k} c_i^2 \frac{n_i}{m_iN_i} A^2$$

where

$$(5.7)\qquad A^2 = \int_0^1 J^2(x)dx - \left(\int_0^1 J(x)dx\right)^2$$

$$(5.8)\qquad \mu(\theta) = E(Q) = \sum_{i=1}^{k} c_i\mu_i(\theta)$$

$$(5.9)\qquad \sigma^2(\theta) = \mathrm{var}(Q) = \sum_{i=1}^{k} c_i^2\sigma_i^2(\theta)$$

where $\mu_i(\theta)$ and $\sigma_i^2(\theta)$ are given by (5.2) and (5.3) respectively.

By the Central Limit Theorem, the distribution of Q will be approximately normal.

It follows that the critical value Q_α is approximately equal to

$$(5.10)\qquad Q_\alpha = \mu(0) + \lambda_\alpha A\sqrt{\sum_{i=1}^{k}(c_i^2n_i)/m_iN_i}$$

where

$$(5.11)\qquad \int_{\lambda_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx = \alpha$$

and the power of the Q test with respect to a given set of distribution functions $F_1(x)$ and $G_1(x)$ is approximately equal to

$$(5.12) \qquad \beta_\alpha(\theta) = 1 - \Phi[(\lambda_\alpha - \frac{\mu(\theta)-\mu(0)}{\sigma(0)}) \frac{\sigma(0)}{\sigma(\theta)}]$$

where $\Phi(x)$ is the standard normal distribution function.

6. Locally Best Q Test

From this section onward, we assume that m_1, n_1 and k are non-decreasing functions of a natural number n that tends to infinity. The dependence on n is indicated when necessary, by writing $m_1(n)$, $n_1(n)$, $k(n)$, $\mu^{(n)}(\theta)$, etc. We shall consider the following two special cases:

Case 1: $m_1(n)$ and $n_1(n)$ tend to infinity as n tends to infinity but $\frac{m_1(n)}{n}$ and $\frac{n_1(n)}{n}$ remain bounded away from zero, $k(n) = k$ for each n.

Case 2: $m_1(n)$ and $n_1(n)$ remain constants and $k(n)$ tends to infinity as n tends to infinity. For simplicity sake, we assume that $m_1(n) = m_1$, $n_1(n) = n_1$ and $k(n) = n$.

Furthermore, we make the following assumption:

Assumption 6.1:

For sufficiently large n,

$$\sqrt{n}[J\left\{H_1(x;n)\right\} - J\left\{F_1(x;n)\right\}]/A$$

remains bounded as n tends to infinity. Then we prove the following

Theorem 6.1.

For each index n, assume the validity of the case 1 and assumption 6.1. Then the Q test with

$$(6.1) \qquad c_i(n) = c\,\frac{d_i(n)m_i(n)N_i(n)}{n_i(n)}$$

where

$$(6.2) \qquad d_i(n) = \int_{-\infty}^{+\infty}[J\{H_i(x;n)\} - J\{F_i(x;n)\}]dF_i(x;n)$$

and c is an arbitrary positive constant, has for $n \to \infty$, asymptotically the largest power against all alternatives for which $d_i(n)$ are positive.

Proof. We shall first prove that

$$\frac{\sigma^{(n)}(0)}{\sigma^{(n)}(\theta)} \to 1 \quad \text{as } n \to \infty.$$

For this it suffices to show that $\sigma^{(n)}(\theta)$ is continuous at $\theta = 0$, uniformly in n. Consider first, the first integral on the right hand side of (5.3) and let it be denoted by $A_i(\theta)$. Thus

$$(6.3) \qquad A_i(\theta) = \iint_{0<x<y<\infty} G_i(x)[1-G_i(y)]J'[H_i(x)]J'[H_i(y)]\,dF_i(x)dF_i(y).$$

Setting $F_i(x) = u$ and $F_i(y) = v$, we rewrite (6.3) as

$$(6.4) \qquad A_i(\theta) = \iint_{0<u<v<1} G_i^*(u)[1-G_i^*(v)]J'[H_i^*(u)]J'[H_i^*(v)]dudv$$

where $G_i^*(u) = G_i[F_i^{-1}(u)]$ and $H_i^*(u) = \lambda_i u + (1-\lambda_i)G_i^*(u)$.

It is clear that integrand is continuous at $\theta = 0$ for almost all u and v.

Furthermore, since

$$G_1^*(u) \leq \frac{1}{1-\lambda_1} H_1^*(u)$$

$$1-G_1^*(v) \leq \frac{1}{1-\lambda_1} [1-H_1^*(v)]$$

$$|J'[H_1^*(u)]| \leq K[H_1^*(u) \left\{ 1-H_1^*(u) \right\}]^{-3/2+\delta}$$

we have, from (6.4)

$$(6.5) \quad |G_1^*(u)[1-G_1^*(v)]J'(H_1^*(u)]J'[H_1^*(v)]|$$
$$\leq K \frac{1}{(1-\lambda_1)^2} [H_1^*(u)]^{-1/2+\delta}[1-H_1^*(v)]^{-1/2+\delta}[H_1^*(v)]^{-3/2+\delta} [1-H_1^*(u)]^{-3/2+\delta} .$$

We may assume $\delta < 1/2$, without loss of generality.

Then, from (6.5)

$$|G_1^*(u)[1-G_1^*(v)]J'[H_1^*(u)]J'[H_1^*(v)]|$$
$$\leq K \frac{1}{(1-\lambda_1)^2} \lambda_1^{-4+4\delta} u^{-1/2+\delta}(1-v)^{-1/2+\delta} v^{-3/2+\delta}(1-u)^{-3/2+\delta}$$

Hence by Cramér ([1957], p. 67) $A_1(\theta)$ is continuous at $\theta = 0$. Similarly the second integral on the right side of (5.3) is continuous at $\theta = 0$. Hence $\sigma_1^{2(n)}(\theta)$ and so a fortiori $\sigma^{2(n)}(\theta)$ is continuous at $\theta = 0$.

Next, because of the assumption (6.1)

$$\frac{\mu^{(n)}(\theta)-\mu^{(n)}(0)}{\sigma^{(n)}(0)} = O(1).$$

Hence the power of the Q test can be approximated by

$$1 - \Phi\left(\lambda_\alpha - \frac{\mu^{(n)}(\theta)-\mu^{(n)}(0)}{\sigma^{(n)}(0)}\right). \tag{6.6}$$

This is maximum, when

$$\frac{\mu^{(n)}(\theta)-\mu^{(n)}(0)}{\sigma^{(n)}(0)} = \frac{\sum_{i=1}^{k} c_i(n)\left[\int_{-\infty}^{+\infty}\left[J\left\{H_i(x;n)\right\} - J\left\{F_i(x;n)\right\}\right] dF_i(x;n)\right.}{A\sqrt{\sum_{i=1}^{k}\left[c_i^2(n)n_i(n)\right]/m_i(n)N_i(n)}} \tag{6.7}$$

is maximum, which is so when $c_i(n)$ is as defined in (6.1)
This completes the proof of the theorem.

7. Computation of $d_i(n)$

The computation of $d_i(n)$ highly depends upon the sequences of alternatives, we have in mind. In subsequent analysis, we shall concern ourselves with three sequences of admissible alternative hypotheses viz. H_n^P, H_n^L and H_n^c. The hypothesis H_n^P specifies that for each $i=1,\ldots,k$; $G_i(x) = F(x+\gamma_i+\frac{\xi}{\sqrt{n}})$, the hypothesis H_n^L specifies that for each $i=1,\ldots,k$; $G_i(x) = [F(x+\gamma_i)]^{1-\frac{\xi}{\sqrt{n}}}$ and the hypothesis H_n^c specifies that for each $i=1,\ldots,k$; $\underline{x_i}$ has the distribution function $F(x+\gamma_i)$ and $\underline{y_i}$ has the distribution function $(1-\frac{\xi}{\sqrt{n}})F(x+\gamma_i) + \frac{\xi}{\sqrt{n}}G(x+\gamma_i)$; where γ_i is a real number, ξ is a finite positive constant independent of i, and $F_i(x) = F(x+\gamma_i)$. Alternatives of the form H_n^P were introduced by Pitman, those of the form H_n^L by Lehmann (1953) in order to study the non-parametric procedures

when the alternatives themselves are given in a non-parametric form. For an extensive study of Lehmann's alternatives, the reader is referred to an interesting paper of Savage (1956). Alternatives of the form H_n^c are referred to as contaminated alternatives, which have been considered by Hodges and Lehmann (1956) among others.

We shall, therefore, compute $d_i(n)$ and hence $c_i(n)$ for the above mentioned classes of alternatives. We shall make use of a lemma due to Hodges and Lehmann (1961) and the reader is referred to this reference regarding it. A consequence of this lemma in a form appropriate for our purpose, may be stated as follows:

Lemma 7.1. (Hodges-Lehmann).

If

(i) F *is continuous cumulative distribution function function differentiable in each of the open intervals* $(-\infty, a_1)$, $(a_1, a_2), \ldots,$ (a_{s-1}, a_s), (a_s, ∞) *and the derivative of* F *is bounded in each of these intervals and either*

(ii) *for the alternative* H_n^P *or* H_n^c *the function* $\frac{dJ[F(x)]}{dx}$ *is bounded as* $x \to \pm\infty$, *or*

(ii') *for the alternatives* H_n^L, *the function* $F(x)\log F(x) \frac{dJ[F(x)]}{dx}$ *is bounded as* $x \to \pm\infty$, *then*

(7.1) $\sqrt{n}\, d_i(n) \sim \xi(1-\lambda_i)\int \frac{dJ[F(x)]}{dx} dF(x)$, *in case the hypothesis* H_n^P *is valid,*

(7.2) $\sqrt{n}\, d_i(n) \sim \xi(1-\lambda_i)\int -F(x) \log F(x)\frac{dJ[F(x)]}{dx} dx$ in case the *hypothesis* H_n^L *is valid, and*

(7.3) $\sqrt{n}\, d_i(n) \sim \xi(1-\lambda_i)\int [G(x)-F(x)]\frac{dJ[F(x)]}{dF(x)} dF(x)$ *in case the hypothesis* H_n^c *is valid.*

The proof of this lemma follows by the method used in section 3 and 4 of Hodges-Lehmann (1961).

In order to save space the details are omitted.

Now the quantities $\xi\int \frac{dJ[F(x)]}{dx} dF(x)$, $\xi\int -F(x) \log F(x) \frac{dJ[F(x)]}{dx}$ and $\int [G(x)-F(x)] \frac{dJ[F(x)]}{dF(x)} dF(x)$ being constants, can be absorbed into the constant c of (6.1), with the result that we have $c_i(n) = cm_i(n)$. Thus the material discussed in this section coupled with the one discussed in the previous section yields the following

Theorem 7.1

For each index n, assume the validity of the hypotheses H_n^P or H_n^L or H_n^C and the assumptions of lemma 7.1. Then for the case 1, the Q test with weights $c_i(n) = cm_i(n)$, where c is an arbitrary positive constant, has asymptotically the maximum power.

In what follows, we shall denote the locally best Wilcoxon form of Q-test by the symbol Q_W and we shall call it locally best Q_W test. Thus

$$Q_W = \sum_{i=1}^{k} cm_i(n) Q_{W_i}$$

where Q_{W_i} is obtained from (4.1) by letting $E_{N,j}^{(i)} = j/N_i$.

8. Relation between Elteren's W test and locally best Q_W test

Let $X_{i,r}$ and $Y_{i,s}$ denote the r^{th} and the s^{th} observations of $\underline{X_i}$ and $\underline{Y_i}$ respectively; $r=1,\ldots,m_i$; $s=1,\ldots,n_i$.

Denote

$$\operatorname{sgn}(X_{i,r}-Y_{i,s}) = \begin{cases} -1 , & \text{if } X_{i,r}-Y_{i,s} < 0 \\ 0 , & \text{if } X_{i,r}-Y_{i,s} = 0 \\ +1 , & \text{if } X_{i,r}-Y_{i,s} > 0 \end{cases}$$

then the Elteren's locally best W test [cf. Elteren, Ph. van (1960)] for case 1 as well as case II is defined as

$$(8.1) \qquad W = c \sum_{i=1}^{k(n)} W_i/[m_i(n)+n_i(n)+1]$$

where

$$(8.2) \qquad W_i = \sum_{r=1}^{m_i} \sum_{s=1}^{n_i} \operatorname{sgn}(X_{i,r}-Y_{i,s})$$

which is equivalent to the Wilcoxon's statistic [cf. Wilcoxon (1945)] for the i^{th} pair of samples.

It is easy to check that

$$(8.3) \qquad W_i = 2m_iN_iQ_{W_i} - m_i(N_i+1)$$

so that

$$(8.4) \qquad W = 2c \sum_{i=1}^{k} \frac{m_iN_i}{m_i+n_i+1} Q_{W_i} - c \sum_{i=1}^{k} m_i .$$

Hence, asymptotically, the following linear relation exists between the W statistic and Q_W statistic.

$$(8.5) \qquad W = 2c\, Q_W - c \sum_{i=1}^{k} m_i .$$

In our subsequent analysis, we shall use the following expressions connected with the Elteren's W test:

$$(8.6) \qquad \mu(\theta) = E(W) = 2c \sum_{i=1}^{k} \frac{m_in_i}{m_i+n_i+1} \int_{-\infty}^{+\infty} [G_i(x)-F_i(x)]dF_i(x)$$

$$(8.7) \qquad \sigma^2(0) = \operatorname{var}_{H_0}(W) = \frac{1}{3} c^2 \sum_{i=1}^{k} \frac{m_in_i}{m_i+n_i+1} \quad .$$

9. On the Combination of Independent Two-Sample tests based on Student's t-statistic.

Let $F_i(x)$ and $G_i(x)$ be normal distribution functions with the same variance σ^2. Then the student's t-test for the i^{th} pair of samples, is based on the statistic

$$(9.1)\qquad t_i = \frac{(\bar{X}_i-\bar{Y}_i)/\sqrt{\frac{1}{m_i}+\frac{1}{n_i}}\,\sigma}{\sqrt{[\sum_{j=1}^{m_i}(X_{ij}-\bar{X}_i)^2+\sum_{k=1}^{n_i}(Y_{ik}-\bar{Y}_i)^2]/(m_i+n_i-2)\sigma^2}}$$

where

$$(9.2)\qquad \bar{X}_i = \sum_{j=1}^{m_i} X_{ij}/m_i \quad\text{and}\quad \bar{Y}_i = \sum_{k=1}^{n_i} Y_{ik}/n_i.$$

But since the denominator of t_i tends to one in probability, therefore an asymptotically equivalent statistic is

$$(9.3)\qquad t_i' = (\bar{X}_i - \bar{Y}_i)/\sqrt{\frac{1}{m_i}+\frac{1}{n_i}}\,\sigma$$

which has normal distribution. Now proceeding as in sections 6 and 7, we conclude

Theorem 9.1.

For each index n, assume the validity of the hypotheses H_n^P, H_n^L or H_n^C. Then the t-test with weights $c_i(n) = c\sqrt{\frac{m_i(n)n_i(n)}{N_i(n)\sigma^2}}$, has for $n\to\infty$, asymptotically the largest power. (c is an arbitrary positive constant).

We may note that

(a) Under H_n^P,

$$(9.4)\qquad \mu_t(\theta) = E(t) = \sum_{i=1}^{k} c\,\frac{m_i(n)n_i(n)}{N_i(n)\sigma^2}\frac{\xi}{\sqrt{n}}$$

(b) Under H_n^L,

$$(9.5)\qquad \mu_t(\theta) = E(t) = -\sum_{i=1}^{k} c\,\frac{m_i(n)n_i(n)}{N_i(n)\sigma^2}\frac{\xi}{\sqrt{n}}\int x(1+\log F(x))\,dF(x)$$

(c) Under H_n^c,

$$(9.6)\qquad \mu_t(\theta) = E(t) = \sum_{i=1}^{k} c\,\frac{m_i(n)n_i(n)}{N_i(n)\sigma^2}\frac{\xi}{\sqrt{n}}\int x\,d(G(x)-F(x))$$

and

$$(9.7)\qquad \sigma_t^2(\theta) = \mathrm{var}(t) = \sum_{i=1}^{k} c^2\,\frac{m_i(n)n_i(n)}{N_i(n)\sigma^2}$$

under H_n^P, H_n^L and H_n^c.

10. The asymptotic relative efficiencies of the tests

Briefly, the idea of the asymptotic relative efficiency is the following:

Suppose that for testing the hypothesis H_o against H_n, two tests T and T^* require N and N^* observations to achieve the same power β at the level of significance α. Then the asymptotic efficiency of T with respect to T^* is defined as

$$\lim_{N\to\infty} N^*/N = e_{T,T^*}(\alpha,\beta,H_o,\{H_n\}).$$

We shall be interested in studying the asymptotic efficiency of (i) the Elteren's W test relative to an arbitrary Q test against (a) Pitman's shift alternatives and (b) Lehmann's distribution free alternatives, and (ii) the Elteren's W test relative to the locally best t test against (a) and (iii) arbitrary Q test relative to the locally best t test against contaminated alternatives.

The asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test is stated in the following

Theorem 10.1(a)

If

(i) for all i, $\lim_{n\to\infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n\to\infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) the distribution function F is such that

$$\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{+\infty} [J \left\{ \lambda_i F_i(x) + (1-\lambda_i) F_i(x + \tfrac{\xi_i}{\sqrt{n}}) \right\} - J \left\{ F_i(x) \right\}] dF_i(x)/A$$

exists,

(iii) the hypothesis of lemma 7.1 are assumed,

then

the asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test for testing the hypothesis H_o against H_n^P is

$$(10.1) \qquad e^P_{W,Q} = 12A^2 \left(\frac{\int_{-\infty}^{+\infty} f^2(x)dx}{\int_{-\infty}^{+\infty} \frac{dJ[F(x)]}{dx} dF(x)} \right)^2$$

where f is the density of F.

Proof.

Let n index the sample size for the Elteren's W test and n* the corresponding index for the Q test. Furthermore, let the level of significance be fixed at α and the limiting power at β. Then the W and Q tests will have the same limiting power, if

$$\frac{2\sum_{i=1}^{k}\frac{m_i(n)n_i(n)}{m_i(n)+n_i(n)+1}\int_{-\infty}^{+\infty}[F(x+\gamma_i+\xi/\sqrt{n})-F(x+\gamma_i)]dF(x+\gamma_i)}{\sqrt{\frac{1}{3}\sum_{i=1}^{k}\frac{m_i(n)n_i(n)}{m_i(n)+n_i(n)+1}}}$$

$$=\frac{\sum_{i=1}^{k} m_i(n^*)[\int_{-\infty}^{+\infty}[J\left\{\frac{m_i(n^*)}{N_i(n^*)}F(x+\gamma_i)+\frac{n_i(n^*)}{N_i(n^*)}F(x+\gamma_i+\xi^*/\sqrt{n^*}\right\} - J\left\{F(x+\gamma_i)\right\}]dF(x+\gamma_i)}{A\sqrt{\sum_{i=1}^{k}\frac{m_i(n^*)n_i(n^*)}{N_i(n^*)}}}$$

i.e. if

$$(10.2)\quad 2\sqrt{3}\sqrt{\sum_{i=1}^{k}\frac{r_i s_i}{(r_i+s_i)}}\ \xi\int_{-\infty}^{+\infty}f(x)dF(x) = \frac{1}{A}\sqrt{\sum_{i=1}^{k}\frac{r_i s_i}{r_i+s_i}}\ \xi^*\int_{-\infty}^{+\infty}\frac{dJ[F(x)]}{dx}dF(x)$$

and the same alternatives, if $\xi/\sqrt{n} = \xi^*/\sqrt{n^*}$.
Substituting $\xi^* = \xi\sqrt{n^*/n}$ in (10.2) yields the desired result.

It may be remarked that (10.1) agrees with the result found by Chernoff-Savage (1958), Hodges-Lehmann (1961) for the two-sample problem and Puri (1962) for the c-sample problem. Hence the efficiency results of this paper as well as those mentioned above apply directly to the present problem.

Proofs of theorems 10.1(b) to 10.1(d) are similar to those of theorem 10.1(a) and are therefore omitted.

<u>Theorem</u> 10.1(b).

<u>If</u>

(i) for all i, $\lim_{n\to\infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n\to\infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) the distribution function F is such that

$$\lim_{n\to\infty} \sqrt{n} \int_{-\infty}^{+\infty} [J \{ \lambda_i F_i(x) + (1-\lambda_i)[F_i(x)]^{1-\xi/\sqrt{n}} \} - J \{ F_i(x) \}] dF_i(x)/A$$

exists,

(iii) the hypotheses of lemma 7.1 are satisfied,

then,

the asymptotic relative efficiency of the Elteren's W test relative to an arbitrary Q test for testing the hypothesis H_o against H_n^L is

$$e^L_{W,Q} = \frac{3}{4} A^2 \left(\frac{1}{\int_{-\infty}^{+\infty} F(x) \log F(x) \frac{dJ}{dF(x)} \{F(x)\} \, dF(x)} \right)^2 . \tag{10.3}$$

In particular when $J = \Phi^{-1}$, where Φ is the cumulative normal distribution function,

$$e^L_{W,Q}(F(x)) = \frac{3}{4} \left(\frac{1}{\int_{-\infty}^{+\infty} x \log \Phi(x) \, d\Phi(x)} \right)^2 \tag{10.4}$$

$= .927$ by numerical evaluation.

We may remark that (10.4) agrees with the result found by Puri (1962) for the c-sample problem.

Theorem 10.1 (c).

If

(i) for all i, $\lim_{n\to\infty} \frac{m_i(n)}{n} = r_i$ and $\lim_{n\to\infty} \frac{n_i(n)}{n} = s_i$ exist and are positive,

(ii) <u>the distribution function</u> F <u>and</u> G <u>are such that</u>

$$\lim_{n\to\infty} \sqrt{n}\int_{-\infty}^{+\infty}[J\left\{\lambda_i F(x+\gamma_i)+(1-\lambda_i)\left\{(1-\theta)F(x+\gamma_i)+\theta\, G(x+\gamma_i)\right\}\right\} - J\left\{F(x+\gamma_i)\right\}]dF(x+\gamma_i)/A$$

<u>exists</u>,

(iii) <u>the hypotheses of lemma</u> 7.1 <u>are assumed</u>,

<u>then</u>,

<u>the asymptotic relative efficiency of an arbitrary</u> Q <u>test relative to locally best</u> T <u>test for testing the hypothesis</u> H_o <u>against</u> H_n^c <u>is</u>

$$(10.5)\qquad e^c_{Q,T}(F,G) = \frac{\sigma^2}{A^2}\left(\frac{\int_{-\infty}^{+\infty}[F(x)-G(x)]\frac{dJ}{dF(x)}\{F(x)\}dF(x)}{\int_{-\infty}^{+\infty}[F(x)-G(x)]dx}\right)^2.$$

If, in particular, $J(u) = u$, then Q test becomes Elteren's W test and we have

$$(10.6)\qquad e^c_{W,T} = 12\,\sigma^2\left(\frac{\int_{-\infty}^{+\infty}[F(x)-G(x)]dF(x)}{\int_{-\infty}^{+\infty}[F(x)-G(x)]dx}\right)^2.$$

We may remark that the result (10.6) agrees with the result found by Hodges, Lehmann (1956) for the two sample problem. Hence their general comments regarding the merits of the performance of Wilcoxon test relative to t test against contaminated alternatives may be carried along the present situation.

<u>Theorem</u> 10.1(d).

<u>If</u>

(i) <u>for all</u> i, $\lim_{n\to\infty}\frac{m_i(n)}{n} = r_i$ <u>and</u> $\lim_{n\to\infty}\frac{n_i(n)}{n} = s_i$ <u>exist and are positive</u>.

(ii) <u>the hypotheses of lemma</u> 7.1 <u>are satisfied,</u>

<u>then,</u>

<u>the asymptotic relative efficiency of the Elteren's</u> W <u>test relative to the locally best</u> T <u>test for testing the hypothesis</u> H_o <u>against</u> H_n^L <u>is</u>

$$(10.7) \qquad e^L_{W,T}(F(x)) = \frac{3}{4}\,\sigma^2 \frac{1}{\left(\int_{-\infty}^{+\infty} x[1+\log F(x)]dF(x)\right)^2}$$

which agrees with the result obtained by the author (1962) for the c-sample problem.

Similarly, it can be shown that

$$(10.8) \qquad e^P_{W,T}(F(x)) = 12\,\sigma^2 \left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2$$

which is known to be the asymptotic efficiency of the two sample Wilcoxon test relative to the student's t test. Hodges and Lehmann (1956) have shown that always $e^P_{W,t}(F(x)) \geq 0.864$. In case F(x) is normal distribution function, this is $3/\pi$.

We now consider the case II. Let $m_i(n) = m_i$, $n_i(n) = n_i$ and suppose that the number, say v, of pair of samples tend to infinity in such a way that the limits:

$$L = \lim_{v\to\infty} \frac{1}{v}\sum_{i=1}^{v} \frac{m_i n_i}{m_i+n_i+1}$$

and

$$M = \lim_{v\to\infty} \frac{1}{v}\sum_{i=1}^{v} \frac{m_i n_i}{m_i+n_i}$$

exist. Then subject to the conditions that underlying distributions satisfy some general regularity conditions, it can be shown that

$$(10.9) \qquad e^P_{W,T}(F(x)) = \frac{12\,\sigma^2 L \left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2}{M} \quad .$$

As a special case, suppose that N repetitions of k blocks are needed for Elteren's locally best W test and N^* for the locally best T test. Then we shall have

$$(10.10) \quad e^P_{W,T}(F(x)) = 12\,\sigma^2 \frac{\sum_{i=1}^{k} \frac{m_i n_i}{m_i+n_i+1}}{\sum_{i=1}^{k} \frac{m_i n_i}{m_i+n_i}} \left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2 \quad .$$

In particular, where $m_i = n_i = 1$

$$(10.11) \qquad e^P_{W,T}(F(x)) = 8\,\sigma^2 \left(\int_{-\infty}^{+\infty} f^2(x)dx\right)^2$$

which is the asymptotic efficiency of the sign test relative to the student's test, a quantity which is usually expressed as

$$(10.12) \qquad e^P_{s,t}(F(x) = 4\,\sigma^2 f^2(0)$$

see in this connection Hodges-Lehmann (1956) and Noether (1958).

It is interesting to note that the asymptotic relative efficiency (10.10) depends on the number of blocks as well as their sizes. When the sample sizes are equal from block to block, say $m_i = n_i = m$, then the asymptotic efficiency (10.10) depends only on the block size 2m. In the special case where F(x) is normal distribution function $\Phi(x)$, we have

(10.13) $$e^{P}_{W,T}(F(x)) = \frac{3}{\pi}\frac{2m}{2m+1}$$

some values of this expression are tabulated below:

(10.14)

m	1	2	3	4	5	6	7	8	9	10	.. ∞
$e^{P}_{W,T}(F)$	.637	.764	.818	.849	.868	.881	.891	.898	.904	.909	.955

In conclusion, we may mention that the results given here are valid for large number of replications.

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Footnotes

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2. Chernoff and Savage use the symbol T_N instead of Q_1.

3. If $\{ x_n \}$ is a sequence of random variables and $\{ r_n \}$ a sequence of positive numbers, we write $x_n = o_p(r_n)$, if x_n/r_n tends to zero in probability, or equivalently, if, for each $\varepsilon > 0$, $P_n \{ |x_n|r_n| \leq \varepsilon \} \rightarrow 1$ as $n \rightarrow \infty$.

Resumé

Sur la combinaison de testes indépendantes d'une classe générale pour duex échantillons.

Dans cet article, on analyse des testes qui sont basées sur des combinaisons linéaires de k statistiques indépendantes pour deux échantillons. On compare deux classes de ces testes, où les statistiques employees sont d'une part du type de Chernoff et Savage [1] et de l'autre part du type de "Student". Sous certaines conditions, on obtient les coéfficients de ces combinaisons linéaires qui donnent les plus grandes puissances asymptotiques locales de ces testes. Ces resultats sont particularisés au cas où les hypothèses alternatives sont les hypothèses (non parametriques) de Pitman ou de Lehmann ou des moyennes pondérées souvent appelées "distributions contaminées". Enfin on discute les efficacités asymptotiques du teste Q relatives à quelques-uns de ses compétiteurs parametriques ainsi que compétiteurs non-paramétriques en relation aux alternatives mentionnées ci-devant.

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